ISSN: 2310-0133
Founder: Academic Publishing House *Researcher*
DOI: 10.13187/issn.2310-0133
Has been issued since 2013.

European Journal of Technology and Design

An Imprecise Model of Combining Expert Judgments About Quantiles

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Abstract. Most models of aggregating expert judgments assume that there is some precise probability distribution characterizing the system behavior and expert information allows us to compute parameters of this distribution. However, judgments elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments, and any assumption concerning a certain distribution in combination with imprecision of judgments may lead to incorrect results. To take into account the imprecision and unreliability of judgments, a model of combining and processing the expert judgments about quantiles of an unknown probability distribution is proposed in this paper. Many results are obtained in the explicit form and are very simple from the computational point of view. Numerical examples illustrate the proposed models.

Keywords: Expert judgments, imprecise probabilities, quantile, uncertainty modelling, natural extension, linear optimization.

1. Introduction

Judgments elicited from human experts may be a very important part of information about systems on which limited experimental observations are possible. Several methods for elicitation, assessment and pooling of this type of information have been proposed in [1, 3, 5, 16, 17]. In order to get useful information from the experts, a proper uncertainty modeling of pieces of data supplied by experts has to be used. As pointed out in [4, 11], the uncertainty models play a central role in the use of expert judgments, because no human being would claim that he is absolutely sure about his judgments or advice. Therefore, it is necessary to incorporate into any model the individual expert's uncertainty about his advice, the decision maker's uncertainty about the quality of the expert(s), and how these two kinds of uncertainty interact and impact on the credibility of the final results.

Judgments elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments. When several experts supply judgments or assessments about a system, their responses are pooled so as to derive a single measure of the system behavior. Most methods of aggregating these assessments assume that there is some precise probability distribution characterizing the system behavior and available expert information allows us to compute parameters of this distribution. However, any assumption concerning a certain distribution in combination with imprecision of expert judgments may lead to incorrect results which often cannot be validated due to the lack of experimental observations. Therefore, it is necessary to aggregate the expert judgments without any assumptions about probability distributions and to use only the information which is available. In order to cope with uncertainty and vagueness of available information, it is proposed to apply the *imprecise probability theory* (also called the theory of lower previsions [14], the theory of interval statistical models [8], the theory of interval probabilities [15]), whose general framework is provided by upper and lower previsions. Some examples of combining the partial and heterogeneous information in reliability analysis by means of the framework of imprecise probabilities can be found in [6, 12, 13].

However, the opinion of reliable experts should be more important than those of unreliable ones. Various methods of the pooling of assessments, taking into account the quality of experts, are available in the literature [3, 9, 10, 16]. These methods use the concept of precise probabilities for modelling the uncertainty. It should be noted that the models of aggregating expert judgments taking into account the quality of experts can be considered in a framework of hierarchical uncertainty models which are rather common in uncertainty theory.

To cope with the lack of precise expert knowledge a framework of the possibility theory has been applied to combining judgments [4, 11]. However, this approach requires assuming a certain type of a possibility distribution to formalize the expert information. Moreover, the obtained results are often too imprecise in order to use them in practice. Therefore, simplified models of combining the common expert judgments about quantiles of an unknown probability distribution are proposed in this paper. Many results are obtained in the explicit form and are very simple from the computational point of view.

It is worth noticing that the considered models of uncertainty differ from standard uncertainty models used in the imprecise probability theory (see section "Preliminary definitions"), where there exists an interval of previsions of a certain gamble. In the models of quantiles, the gamble can be viewed as a set of gambles for which the same previsions are defined. Various numerical examples illustrate the proposed models.

2. Preliminary definitions

Suppose there is a discrete random variable X defined on the sample space Ω and information about this variable is represented as a set of m interval-valued expectations of functions $f_1(X),\dots,f_m(X)$. Denote these lower and upper expectations $\underline{a}_i=\underline{\mathbf{E}}f_i$ and $\overline{a}_i=\overline{\mathbf{E}}f_i$, $i=1,\dots,m$. In terms of the theory of imprecise probabilities the corresponding functions $f_i(X)$ and interval-valued expectations $\underline{\mathbf{E}}f_i$ and $\overline{\mathbf{E}}f_i$, $i=1,\dots,m$, are called *gambles* and *lower and upper previsions*, respectively. We can model various types of information by means of lower and upper previsions. For example, if f_i is the indicator function of an event A, then previsions $\underline{\mathbf{E}}f_i$ and $\overline{\mathbf{E}}f_i$ can be regarded as lower and upper probabilities of the event A. If $f_i(X)=X$, then $\underline{\mathbf{E}}X$ and $\overline{\mathbf{E}}X$ are bounds for the mean value of the corresponding random variable. The lower and upper previsions $\underline{\mathbf{E}}f_i$ and $\overline{\mathbf{E}}f_i$ can be regarded as bounds for an unknown precise prevision $\mathbf{E}f_i$ which will be called a *linear prevision*.

For computing new previsions $\underline{\mathbf{E}}g$ and $\overline{\mathbf{E}}g$ of a gamble $g(X)$ from the available information, *natural extension* can be used. Natural extension is a general mathematical procedure for calculating new previsions from initial judgments. It produces a coherent overall model from a certain collection of imprecise probability judgments and may be seen as the basic constructive step in interval-valued statistical reasoning. It is written as the following optimization problems:

$$\underline{\mathbf{E}}g=\min_p \sum_{x\in\Omega} g(x)p(x),\ \overline{\mathbf{E}}g=\max_p \sum_{x\in\Omega} g(x)p(x), \tag{1}$$

subject to

$$p(x)\geq 0,\ \sum_{x\in\Omega} p(x)=1,\ \underline{a}_i\leq \sum_{x\in\Omega} f_i(x)p(x)\leq \overline{a}_i,\ i\leq m. \tag{2}$$

Here the minimum and maximum are taken over a set of all possible probability distributions $\{p(x)\}$ satisfying conditions (2).

Optimization problems (1)-(2) can be explained as follows. The linear prevision $\mathbf{E}g$ can be computed as

$$\mathbf{E}g=\sum_{x\in\Omega} g(x)p(x).$$

However, we do not know the distribution p because our initial information is restricted only by the lower and upper previsions $\underline{\mathbf{E}}f_i$ and $\overline{\mathbf{E}}f_i$, $i=1,...,m$, and there is no information about distributions of X. At the same time, the available lower and upper previsions produce the set of possible distributions that are consistent with these previsions. This means that we can find only the largest and smallest possible values of $\mathbf{E}g$ for all distributions from the set $\{p(x)\}$. It can be carried out by solving optimization problems (1)-(2).

It should be noted that problems (1)-(2) are linear and the dual optimization problems can be written as follows [8,13]:

$$\overline{\mathbf{E}}g = \min_{c_0,c_i,d_i}\left(c_0+\sum_{i=1}^{m}\left(c_i\overline{a}_i - d_i\underline{a}_i\right)\right),\ \underline{\mathbf{E}}g = -\overline{\mathbf{E}}(-g), \tag{3}$$

subject to c_i, $d_i \in \mathbf{R}_+$, $c_0 \in \mathbf{R}$, $i=1,...,m$, and $\forall x \in \Omega$, (4)

$$c_0+\sum_{i=1}^{m}\left(c_i-d_i\right)f_i(x) \geq g(x).$$

Here c_0, c_i, d_i are the optimization variables such that c_0 corresponds to the constraint $\sum_{x\in\Omega}p(x)=1$, c_i corresponds to the constraint $\sum_{x\in\Omega}f_i(x)p(x) \leq \overline{\mathbf{E}}f_i$, and d_i corresponds to the constraint $\underline{\mathbf{E}}f_i \leq \sum_{x\in\Omega}f_i(x)p(x)$. The optimization variables c_i, d_i, $i=1,...,m$, are defined on all positive real number $\mathbf{R}_+$, the variable c_0 is defined on the set of all real numbers $\mathbf{R}$. It turns out that in many applications the dual optimization problems are simpler in comparison with problems (1)-(2) because this representation allows avoiding the situation when a number of the optimization variables is infinite. Of course, the dual optimization problems have generally an infinite number of constraints each of them is defined by a value of x. However, as it will be shown below, the number of constraints can be reduced to a finite number.

3. The problem statement

In the probabilistic approach, experts are typically asked about quantiles of a random variable X defined on a sample space Ω. The smallest number $x \in \Omega$, such that $\Pr\{X \leq x\} = k/100$, is called the $k\%$ quantile and denoted $qk\%$. In this approach the experts are often asked to supply the 5%, 50% and the 95% quantiles. In other words, an expert supplies x_1, x_2 and x_3 such that $\Pr\{X \leq x_1\}=0.05$, $\Pr\{X \leq x_2\}=0.5$ and $\Pr\{X \leq x_3\}=0.95$, respectively. Based on these values, and on the choice of a parameterized family of distribution functions, a fitted distribution function is chosen that represents the available information in some best way to some extent.

Suppose that n experts provide their judgments about q_i, $i=1,...,n$, quantiles of an unknown cumulative discrete probability distribution of the random variable X. In particular, the experts may provide judgments about one q quantile, i.e., $q_i = q$, $\forall i \leq n$. This information can be represented as

$$\Pr\{X \leq x_i\} = q_i,\ i=1,...,n,$$

where points x_i, $i=1,...,n$, are elicited from experts.

In terms of the imprecise probability theory, the probability q_i can be viewed as identical lower and upper previsions of the gamble $I_{[0,x_i]}(X)$, i.e., $\underline{\mathbf{E}}I_{[0,x_i]}(X)=\overline{\mathbf{E}}I_{[0,x_i]}(X)$. We assume that the sample space $\Omega=\{x_0,x_1,...,x_N\}$ is discrete and finite for simplicity. As pointed out in [4], experts better supply intervals rather than point-values because their knowledge is not only of limited reliability, but also imprecise. In other words, experts provide some intervals of quantiles in the form $\mathbf{X}_i=[\underline{x}_i,\overline{x}_i]$. This information can be formally written as

$$\Pr\{X \leq [\underline{x}_i,\overline{x}_i]\} = q_i,\ i=1,...,n.$$

It can be seen from Fig.1 that this interval produces a set of probability distributions such that a lower distribution contains the point $q(\overline{x}_i)$ and the upper one contains the point $q(\underline{x}_i)$. Now the question arises. How to interpret the intervals of quantiles? It depends on experts, i.e., on their imagination of interval quantiles. Two models of the expert imagination can be marked out. The first model corresponds to the expert judgment: "I do not know exactly the true value of the quantile, but one of the values in the interval $[\underline{x}_i, \overline{x}_i]$ is true". The second model corresponds to the expert judgment: "All points in the interval $[\underline{x}_i, \overline{x}_i]$ are true values of the quantile". Of course, the first model is more common in practice of elicitation of judgments from experts. Therefore, we will deal with the imprecise judgments by considering them from the first model point of view.

Our aim is to find parameters of an unknown distribution of X, which can be represented as expectations $\mathbf{E}g$ of some functions $g(X)$.

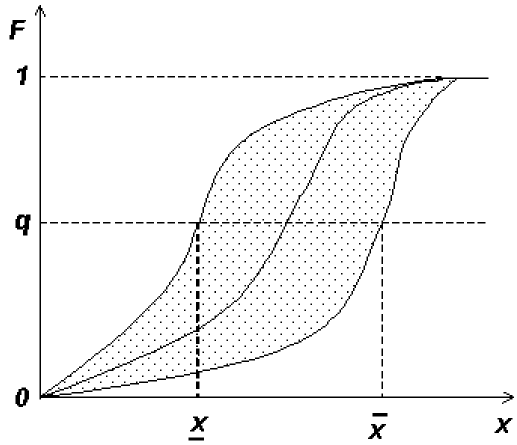

Fig. 1. An example of distributions corresponding to the interval-valued quantile

4. First-order model

Suppose that we know precise values of q_i quantiles t_i, $i=1,...,n$. Denote $\mathbf{T}=(t_1,...,t_n)$ and let $\underline{\mathbf{E}}g\{\mathbf{T}\}$ and $\overline{\mathbf{E}}g\{\mathbf{T}\}$ be lower and upper previsions of the function g under condition of precise values t_i of quantiles. By using the natural extension for computing the lower prevision of the function g, we get the following linear programming problem:

$$\underline{\mathbf{E}}g\{\mathbf{T}\}=\max_{c_0,w_i}\left(c_0+\sum_{i=1}^{n}w_i q_i\right), \qquad (5)$$

subject to $w_i \in \mathbf{R}$, $c_0 \in \mathbf{R}$, $i=1,...,n$, and $\forall x \in \Omega$,

$$c_0+\sum_{i=1}^{n}w_i I_{[0,t_i]}(x) \le g(x). \qquad (6)$$

Here w_i, $i=1,...,n$, are optimization variables obtained by replacing variables c_i and d_i in (3)-(4) due to the equality $\overline{a}_i=\underline{a}_i=q_i$, i.e., $w_i=c_i-d_i$. The upper prevision $\overline{\mathbf{E}}g\{\mathbf{T}\}$ is computed as

$$\overline{\mathbf{E}}g\{\mathbf{T}\}=\min_{c_0,w_i}\left(c_0+\sum_{i=1}^{n}w_i q_i\right), \qquad (7)$$

subject to $w_i \in \mathbf{R}$, $c_0 \in \mathbf{R}$, $i=1,...,n$, and $\forall x \in \Omega$,

$$c_0+\sum_{i=1}^{n}w_i I_{[0,t_i]}(x) \ge g(x). \qquad (8)$$

These optimization problems can be easily derived from (3)-(4).

Proposition 1. Suppose that $q_1 \le q_2 \le ... \le q_n$. Denote $q_0 = 0$, $q_{n+1} = 1$, $t_0 = x_0$, and $t_{n+1} = x_N$. Then solutions to problems (5)-(6) and (7)-(8) exist if $t_1 \le t_2 \le ... \le t_n$. Denote

$$\underline{\mathbf{E}}\widehat{g}\{\mathbf{T}\} = g(x_0)q_1 + \sum_{i=1}^{n} g(t_i)(q_{i+1} - q_i), \qquad (9)$$

$$\overline{\mathbf{E}}\widehat{g}\{\mathbf{T}\} = g(x_N)(1 - q_n) + \sum_{i=1}^{n} g(t_i)(q_i - q_{i-1}). \qquad (10)$$

If the function $g(x)$ is non-decreasing, then $\underline{\mathbf{E}}g\{\mathbf{T}\} = \underline{\mathbf{E}}\widehat{g}\{\mathbf{T}\}$, $\overline{\mathbf{E}}g\{\mathbf{T}\} = \overline{\mathbf{E}}\widehat{g}\{\mathbf{T}\}$. If the function $g(x)$ is non-increasing, then $\underline{\mathbf{E}}g\{\mathbf{T}\} = \overline{\mathbf{E}}\widehat{g}\{\mathbf{T}\}$, $\overline{\mathbf{E}}g\{\mathbf{T}\} = \underline{\mathbf{E}}\widehat{g}\{\mathbf{T}\}$.

Proof: First, we consider the case of the non-decreasing function g. It is obvious that any solution exists if $t_1 \le t_2 \le ... \le t_n$ because the judgments $\Pr\{X \le t_1\} = q_1$ and $\Pr\{X \le t_2\} = q_2$ by $q_1 \le q_2$ and $t_1 > t_2$ are conflicting and inconsistent. Let $t_1 \le t_2 \le ... \le t_n$. Let us divide the interval $[0, x_N]$ into $n+1$ adjacent subintervals $[t_i, t_{i+1}]$, $i = 0,...,n$. Since the function g is non-decreasing, then constraints to problem (5)-(6) can be rewritten as

$$\begin{aligned}
c_0 + w_1 + w_2 + ... + w_n &\le g(t_0), \\
c_0 + w_2 + ... + w_n &\le g(t_1), \\
c_0 + w_3 + ... + w_n &\le g(t_2), \\
&\cdots \\
c_0 &\le g(t_n).
\end{aligned}$$

Let us prove that the optimal solution to problem (5)-(6) is $c_0 = g(t_n)$, $w_i = g(t_{i-1}) - g(t_i)$, $i = 2,...,n$, $w_1 = g(t_0) - g(t_1)$. This solution satisfies all constraints. Let us consider the first and the last constraints. Denote $\sum_{i=1}^{n} w_i = W$ and $\sum_{i=1}^{n} w_i q_i = WQ$. It is obvious that $Q \le 1$. Let us rewrite (5)-(6) as follows:

$$\underline{\mathbf{E}}g\{\mathbf{T}\} = \max_{c_0, W} \left(c_0 + WQ\right),$$

subject to $W \in \mathbf{R}$, $c_0 \in \mathbf{R}$, and

$$c_0 + W \le g(t_0), \; c_0 \le g(t_n).$$

If an optimal solution to optimization problem (5)-(6) exists, then it will be a part of optimal solutions to the above problem because some of the constraints to the initial problem were removed. This problem has the following solution: $c_0 = g(t_n)$, $W = g(t_0) - g(t_n)$. It follows from the n -th constraint to problem (5)-(6) that $w_n = g(t_{n-1}) - c_0 = g(t_{n-1}) - g(t_n)$. It follows from the $(n-1)$ -th constraint that $w_{n-1} = g(t_{n-2}) - c_0 - w_n = g(t_{n-2}) - g(t_{n-1})$. By continuing the determination of optimal values of w_i, we get the optimal solution. By assuming $t_0 = x_0$, there holds

$$\begin{aligned}
\underline{\mathbf{E}}g\{\mathbf{T}\} &= g(t_n) - \sum_{i=1}^{n} (g(t_{i-1}) - g(t_i))q_i \\
&= g(x_0)q_1 + \sum_{i=1}^{n} g(t_i)(q_{i+1} - q_i).
\end{aligned}$$

The upper bound is similarly determined. In this case, the constraints to problem (7-(8) are

$$c_0 + w_1 + w_2 + \ldots + w_n \geq g(t_1),$$
$$c_0 + w_2 + \ldots + w_n \geq g(t_2),$$
$$c_0 + w_3 + \ldots + w_n \geq g(t_3),$$
$$\ldots$$
$$c_0 \geq g(t_{n+1}),$$

and the optimal solution is $c_0 = g(t_{n+1}) = g(x_N)$, $w_i = g(t_i) - g(t_{i+1})$, $i = 1,\ldots,n-1$, $w_n = g(t_n) - g(x_N)$. A case of the non-increasing function g is similarly proved.

Since at least one of the points t_i belonging to the interval $[\underline{x}_i, \overline{x}_i]$ is a true value of the corresponding quantile, then there hold

$$\underline{\mathbf{E}}g = \min_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\ldots,n} \underline{\mathbf{E}}g\{\mathbf{T}\},\ \overline{\mathbf{E}}g = \max_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\ldots,n} \overline{\mathbf{E}}g\{\mathbf{T}\}.$$

Proposition 2. Suppose that $t_i \in [\underline{x}_i, \overline{x}_i]$, $i = 1,\ldots,n$. If there exist such $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,n\}$ that $\underline{x}_i > \overline{x}_j$ and $q_i < q_j$, then expert judgments are conflicting. Let

$$\underline{\mathbf{E}}\widehat{g} = g(x_0)q_1 + \sum_{i=1}^{n} g\left(\max_{k=1,\ldots,i} \underline{x}_k\right)(q_{i+1} - q_i), \quad (11)$$

$$\overline{\mathbf{E}}\widehat{g} = g(x_N)(1 - q_n) + \sum_{i=1}^{n} g\left(\min_{k=i,\ldots,n} \overline{x}_k\right)(q_i - q_{i-1}). \quad (12)$$

If the function $g(x)$ is non-decreasing, then $\underline{\mathbf{E}}g = \underline{\mathbf{E}}\widehat{g}$, $\overline{\mathbf{E}}g = \overline{\mathbf{E}}\widehat{g}$. If the function $g(x)$ is non-increasing, then $\underline{\mathbf{E}}g = \overline{\mathbf{E}}\widehat{g}$, $\overline{\mathbf{E}}g = \underline{\mathbf{E}}\widehat{g}$.

Proof: If there hold $\underline{x}_i > \overline{x}_j$ and $q_i < q_j$ for any i and j, then it is impossible to find points t_i and t_j satisfying the condition $t_i \leq t_j$. Consider the case of the non-decreasing function g. Then, according to Proposition 1, we can write

$$\underline{\mathbf{E}}g = \min_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\ldots,n} \underline{\mathbf{E}}g\{\mathbf{T}\} = g(x_0)q_1 + \min_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\ldots,n} \sum_{i=1}^{n} g(t_i)(q_{i+1} - q_i).$$

In order to achieve the minimum, it is necessary to take minimal values of t_i, $i = 1,\ldots,n$. These values are $\underline{x}_i$. However, there is the additional condition of consistency $t_1 \leq t_2 \leq \ldots \leq t_n$. Let $t_1 = \underline{x}_1$. In order to satisfy the condition of consistency, we have to take $t_2 = \max(t_1, \underline{x}_2) = \max(\underline{x}_1, \underline{x}_2)$. By continuing the determination of minimal values of t_i, we get $t_i = \max_{k=1,\ldots,i} \underline{x}_k$. After substituting the optimal values of t_i into the objective function, we get $\underline{\mathbf{E}}g$. The upper prevision is similarly proved. Let $t_n = \overline{x}_n$. In order to satisfy the condition of consistency, we have to take $t_{n-1} = \min(t_n, \overline{x}_{n-1}) = \min(\overline{x}_n, \overline{x}_{n-1})$. By continuing the determination of maximal values of t_i, we get $t_i = \min_{k=i,\ldots,n} \overline{x}_k$. The case of the non-increasing function g is similarly proved.

Corollary 1. If $q_1 = q_2 = \ldots = q_n = q$, then

$$\underline{\mathbf{E}}\widehat{g} = qg(x_0) + (1-q)g\left(\max_{k=1,\ldots,n} \underline{x}_k\right),$$

$$\overline{\mathbf{E}}\widehat{g} = (1-q)g(x_N) + qg\left(\min_{k=1,\ldots,n} \overline{x}_k\right).$$

If the function $g(x)$ is non-decreasing, then $\underline{\mathbf{E}}g = \underline{\mathbf{E}}\widehat{g}$, $\overline{\mathbf{E}}g = \overline{\mathbf{E}}\widehat{g}$. If the function $g(x)$ is non-increasing, then $\underline{\mathbf{E}}g = \overline{\mathbf{E}}\widehat{g}$, $\overline{\mathbf{E}}g = \underline{\mathbf{E}}\widehat{g}$.

Proof: The proof follows directly from Proposition 2.

Let us consider some important special cases of the function g.

Suppose that $g(X) = I_{(\tau, x_N]}(X)$ (the corresponding lower and upper previsions are values of a survival function of X at point τ, i.e., $\Pr\{X > \tau\}$). The function $g(X) = I_{(\tau, x_N]}(X)$ is non-decreasing and takes two values 0 and 1. Then there hold

$$\underline{\mathbf{E}}I_{[\tau, x_N]}(X) = \sum_{i=1}^{n}(q_{i+1} - q_i) I_{(\tau, x_N]}\left(\max_{k=1,\dots,i} \underline{x}_k\right),$$

$$\overline{\mathbf{E}}I_{[\tau, x_N]}(X) = (1 - q_n) + \sum_{i=1}^{n}(q_i - q_{i-1}) I_{(\tau, x_N]}\left(\min_{k=i,\dots,n} \overline{x}_k\right).$$

If $q_1 = \dots = q_n = q$, then

$$\underline{\mathbf{E}}I_{[\tau, x_N]}(X) = (1 - q) I_{(\tau, x_N]}\left(\max_{k=1,\dots,i} \underline{x}_k\right),$$

$$\overline{\mathbf{E}}I_{[\tau, x_N]}(X) = (1 - q) + q I_{(\tau, x_N]}\left(\min_{k=i,\dots,n} \overline{x}_k\right).$$

Suppose that $g(X) = I_{[0,\tau]}(X)$ (the corresponding lower and upper previsions are values of a cumulative distribution function of X at the point τ, i.e., $\Pr\{X \le \tau\}$). The function $g(X) = I_{[0,\tau]}(X)$ is non-increasing and takes two values 0 and 1. Then there hold

$$\underline{\mathbf{E}}I_{[0,\tau]}(X) = (1 - q_n) + \sum_{i=1}^{n}(q_i - q_{i-1}) I_{[0,\tau]}\left(\min_{k=i,\dots,n} \overline{x}_k\right),$$

$$\overline{\mathbf{E}}I_{[0,\tau]}(X) = q_1 + \sum_{i=1}^{n}(q_{i+1} - q_i) I_{[0,\tau]}\left(\max_{k=1,\dots,i} \underline{x}_k\right).$$

If $q_1 = \dots = q_n = q$, then

$$\underline{\mathbf{E}}I_{[0,\tau]}(X) = (1 - q) + q I_{[0,\tau]}\left(\min_{k=i,\dots,n} \overline{x}_k\right),\ \overline{\mathbf{E}}I_{[0,\tau]}(X) = q + (1 - q) I_{[0,\tau]}\left(\max_{k=1,\dots,i} \underline{x}_k\right).$$

Since the function $g(X) = X^m$ is non-decreasing, then the upper and lower m -th moments of X are determined as

$$\underline{\mathbf{E}}X^m = x_0^m q_1 + \sum_{i=1}^{n}\left(\max_{k=1,\dots,i} \underline{x}_k\right)^m (q_{i+1} - q_i),$$

$$\overline{\mathbf{E}}X^m = x_N^m (1 - q_n) + \sum_{i=1}^{n}\left(\min_{k=i,\dots,n} \overline{x}_k\right)^m (q_i - q_{i-1}).$$

If $q_1 = \dots = q_n = q$, then there hold

$$\underline{\mathbf{E}}X^m = x_0^m q + \left(\max_{k=1,\dots,n} \underline{x}_k\right)^m (1 - q),\ \overline{\mathbf{E}}X^m = x_N^m (1 - q) + \left(\min_{k=i,\dots,n} \overline{x}_k\right)^m q.$$

Corollary 2. For any point $\tau \in \Omega$, there hold $\underline{\mathbf{E}}I_{[\tau, x_N]}(X) + \overline{\mathbf{E}}I_{[0,\tau]}(X) = 1$ and $\overline{\mathbf{E}}I_{[\tau, x_N]}(X) + \underline{\mathbf{E}}I_{[0,\tau]}(X) = 1$.

Proof: Let us consider the first equality. Note that if the inequality $\min_{k=i,\dots,n} \overline{x}_k > \tau$ is valid, then the inequality $\min_{k=i,\dots,n} \overline{x}_k \leq \tau$ is violated and vice versa. Let J be a subset of indices $i = 1,\dots,n$, for which the inequality $\min_{k=i,\dots,n} \overline{x}_k > \tau$ is valid. Then

$$\underline{\mathbf{E}}I_{[\tau, x_N]}(X) + \overline{\mathbf{E}}I_{[0,\tau]}(X) = \sum_{i\in J}(q_{i+1} - q_i) + q_1 + \sum_{i\notin J}(q_{i+1} - q_i)$$

$$= q_1 + \sum_{i=1}^{n}(q_{i+1} - q_i) = q_{n+1} = 1,$$

as was to be proved. The second equality is similarly proved.

It follows from Corollary 2 that the property of coherence of imprecise probabilities is fulfilled for the proposed model.

Proposition 3. Suppose that $q_i = Q_1$ for any $i = 1,\dots,n_1$, $q_i = Q_2$ for any $i = n_1+1,\dots,n_1+n_2,\dots$, $q_i = Q_m$, for any $i = n_1 + \dots + n_{m-1} + 1,\dots,n$, i.e., there are m groups of quantiles containing n_i identical values of q_i, $i = 1,\dots,m$. Then

$$\underline{\mathbf{E}}\widehat{g} = g(x_0)Q_1 + \sum_{i=1}^{m} g\left(\max_{k=1,\dots,i} \underline{z}_k\right)(Q_{i+1} - Q_i),$$

$$\overline{\mathbf{E}}\widehat{g} = g(x_N)(1 - Q_m) + \sum_{i=1}^{m} g\left(\min_{k=i,\dots,m} \overline{z}_k\right)(Q_i - Q_{i-1}),$$

where $\underline{z}_k = \max_{l=n_1+\dots+n_{k-1}+1,\dots,n_1+\dots+n_k} \underline{x}_l$, $\overline{z}_k = \min_{l=n_1+\dots+n_{k-1}+1,\dots,n_1+\dots+n_k} \overline{x}_l$

Proof: Let us consider the case of $m = 2$ for simplicity. Then there holds

$$\underline{\mathbf{E}}\widehat{g} = g(x_0)q_1 + \sum_{i=1}^{n_1} g\left(\max_{k=1,\dots,i} \underline{x}_k\right)(q_{i+1} - q_i) + \sum_{i=n_1+1}^{n_1+n_2} g\left(\max_{k=1,\dots,i} \underline{x}_k\right)(q_{i+1} - q_i)$$

$$= g(x_0)q_1 + g\left(\max_{k=1,\dots,n_1} \underline{x}_k\right)(Q_2 - Q_1) + g\left(\max_{k=1,\dots,n_1+n_2} \underline{x}_k\right)(1 - Q_2).$$

Note that

$$\max_{k=1,\dots,n_1} \underline{x}_k = \underline{z}_1, \quad \max_{k=1,\dots,n_1+n_2} \underline{x}_k = \max(\underline{z}_1, \underline{z}_2).$$

Hence

$$\underline{\mathbf{E}}\widehat{g} = g(x_0)q_1 + \sum_{i=1}^{2} g\left(\max_{k=1,\dots,i} \underline{z}_k\right)(Q_{i+1} - Q_i),$$

as was to be proved. The upper bound $\overline{\mathbf{E}}\widehat{g}$ is similarly proved.

Proposition 3 describes a property of decomposition of judgments. A number of experts often provide judgments about the same quantile. In this case, these judgments can be easily aggregated by computing a maximal value of lower bounds $\underline{x}_l$ and a minimal value of upper bounds $\overline{x}_l$. In fact, the obtained interval $[\underline{z}_k, \overline{z}_k]$ is none other than the intersection of intervals corresponding to the identical probabilities q_i.

Let us consider general properties of the proposed model.

Proposition 4. If judgments about quantiles are non-conflicting, then the following properties of coherent previsions hold:

(i) $\underline{\mathbf{E}}g \leq \overline{\mathbf{E}}g$,

(ii) $\underline{\mathbf{E}}g = -\overline{\mathbf{E}}(-g)$,

(iii) $\underline{\mathbf{E}}(a+bg) = a + b\underline{\mathbf{E}}g$, $\overline{\mathbf{E}}(a+bg) = a + b\overline{\mathbf{E}}g$, $\forall b \in \mathbf{R}^{+}$, $\forall a \in \mathbf{R}$,

(iiii) if $\forall x \in \Omega,\ g(x) \le f(x)$, then $\underline{\mathbf{E}}g \le \underline{\mathbf{E}}f$ and $\overline{\mathbf{E}}g \le \overline{\mathbf{E}}f$.

Proof: It is known that the natural extension produces coherent previsions, i.e., it follows from optimization problems (5)-(8) that properties (i), (ii), (iii), (iiii) are valid for $\underline{\mathbf{E}}g\{\mathbf{T}\}$ and $\overline{\mathbf{E}}g\{\mathbf{T}\}$. Then it follows from the inequality $\underline{\mathbf{E}}g\{\mathbf{T}\} \le \overline{\mathbf{E}}g\{\mathbf{T}\}$ that

$$\underline{\mathbf{E}}g = \min_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\dots,n} \underline{\mathbf{E}}g\{\mathbf{T}\} \le \max_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\dots,n} \underline{\mathbf{E}}g\{\mathbf{T}\}$$

$$\le \max_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\dots,n} \overline{\mathbf{E}}g\{\mathbf{T}\} = \overline{\mathbf{E}}g.$$

It follows from the equality $\underline{\mathbf{E}}g\{\mathbf{T}\} = -\overline{\mathbf{E}}(-g\{\mathbf{T}\})$ that

$$\underline{\mathbf{E}}g = \min_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\dots,n} \underline{\mathbf{E}}g\{\mathbf{T}\} = \min_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\dots,n} \left(-\overline{\mathbf{E}}(-g\{\mathbf{T}\})\right)$$

$$= -\max_{t_i \in [\underline{x}_i, \overline{x}_i],\ i=1,\dots,n} \overline{\mathbf{E}}(-g\{\mathbf{T}\}) = -\overline{\mathbf{E}}(-g),$$

The third and fourth properties are similarly proved.

Example 1. Suppose three experts provide their judgments about 5%, 50%, and 95% quantiles of a probability distribution of a random variable X defined on the sample space $\Omega = \{0,1,\dots,100\}$. Their judgments are given in Table 1. Let us find lower and upper expectations ($\underline{\mathbf{E}}X$, $\overline{\mathbf{E}}X$) and probability distributions ($\underline{\mathbf{E}}I_{[0,\tau]}(X), \overline{\mathbf{E}}I_{[0,\tau]}(X)$, $\tau \in \Omega$) of the random variable X . By using Proposition 2, we get $\underline{\mathbf{E}}X = 34.2$, $\overline{\mathbf{E}}X = 77.25$. The lower $\underline{F}(x)$ and upper $\overline{F}(x)$ probability distributions are depicted in Fig.2.

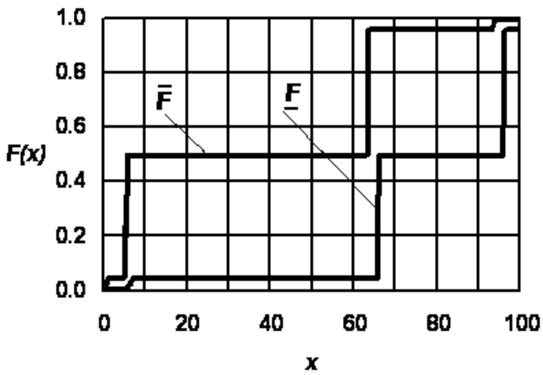

Fig. 2. Lower and upper probability distributions elicited from three experts

Table 1. Expert judgments about 5%, 50%, and 95% quantiles (L-lower, U-upper)

E xpert	Quantiles					
	5%		50 %		95 %	
1		0	4	0	5	5
2			2	0	6	6
3			8	5	0	5

It is interesting to note that imprecision of results in Example 1 depends mainly on a number of types of quantiles, for which experts provide their judgments. At the same time, a number of experts does not influence significantly on the precision. This fact is illustrated by the following examples.

Example 2. Let us add additional judgments about 35% quantile (see Table 2) to the available ones given in Example 1. Then the lower and upper expectations of X are $\underline{\mathbf{E}}X = 37.8$, $\overline{\mathbf{E}}X = 67.95$. The lower $\underline{F}(x)$ and upper $\overline{F}(x)$ probability distributions are depicted in Fig. 3.

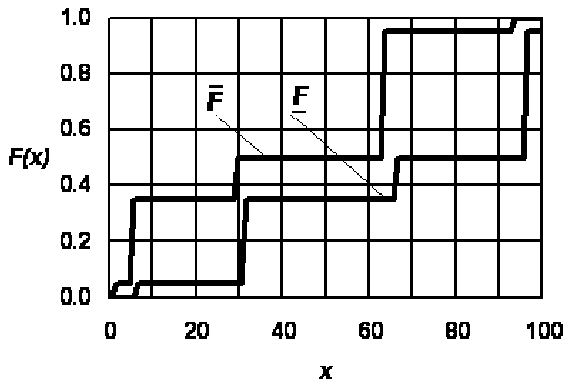

Fig. 3. Lower and upper probability distributions by additional judgments

It can be seen from Example 2 that the imprecision of obtained results is significantly reduced in comparison with results of Example 1.

Example 3. Let us add additional judgments elicited from the fourth expert about 5%, 50%, and 95% quantiles (see Table 3) to the available judgments given in Example 1. Then the lower and upper expectations of X are $\underline{\mathbf{E}}X = 34.75$, $\overline{\mathbf{E}}X = 75.9$. The lower $\underline{F}(x)$ and upper $\overline{F}(x)$ probability distributions are almost the same as ones depicted in Fig.2.

Table 2. Additional expert judgments about 35% quantile (L-lower, U-upper)

Expert	Quantiles	
	35%	
	L	U
1	26	40
2	20	34
3	28	35

Table 3. Additional expert judgements about 5%, 50%, and 95% quantiles (L-lower, U-upper)

Expert	Quantiles					
	5%		50%		95%	
	L	U	L	U	L	U
4	5	6	60	63	92	94

Example 3 shows that the fourth expert does not reduce the available imprecision essentially, but, in any case, it was reduced. The following proposition states this fact.

Proposition 5. The lower bound $\underline{\mathbf{E}}g$ does not decrease and the upper bound $\overline{\mathbf{E}}g$ does not increase by adding arbitrary non-conflicting judgments.

Proof: Let us consider expressions (11)-(12). Suppose that the function g is non-decreasing. The increasing of the lower bound is obvious. Therefore, we prove the decreasing of the upper bound. Suppose that we get an additional $(n+1)$-th judgment $\Pr\{X \leq [\underline{x}_{n+1}, \overline{x}_{n+1}]\} = q_{n+1}$. Without loss of generality, we assume that $q_{n+1} \geq q_n$. Let $\overline{\mathbf{E}}^{(n)}g$ and $\overline{\mathbf{E}}^{(n+1)}g$ be upper previsions obtained by the given n and $n+1$ judgments, respectively. Then

$$\overline{\mathbf{E}}^{(n)}g - \overline{\mathbf{E}}^{(n+1)}g = g(x_N)(1-q_n) + \sum_{i=1}^{n} g\left(\min_{k=i,\dots,n} \overline{x}_k\right)(q_i - q_{i-1})$$

$$-g(x_N)(1-q_{n+1}) - \sum_{i=1}^{n+1} g\left(\min_{k=i,\dots,n+1} \overline{x}_k\right)(q_i - q_{i-1})$$

$$= \left(g(x_N) - g(\overline{x}_{n+1})\right)(q_{n+1} - q_n)$$

$$+\sum_{i=1}^{n}\left(g\left(\min_{k=i,\dots,n} \overline{x}_k\right) - g\left(\min_{k=i,\dots,n+1} \overline{x}_k\right)\right) \times (q_i - q_{i-1}) \geq 0,$$

because the inequalities $\min_{k=i,\dots,n} \overline{x}_k \geq \min_{k=i,\dots,n+1} \overline{x}_k$, $q_i \geq q_{i-1}$, and $x_N \geq \overline{x}_{n+1}$ are valid. This implies that $\overline{\mathbf{E}}^{(n)}g \geq \overline{\mathbf{E}}^{(n+1)}g$, as was to be proved. The proof for the non-decreasing function g is similar.

5. Conclusion

The first-order model of aggregating expert judgments about imprecise quantiles has been proposed in the paper. The main virtue of the model is that it does not use information about a probability distribution of the considered random variable. Of course, this feature leads to imprecise results which are represented in the form of intervals of previsions. At the same time, the risk of possible errors in this case is reduced. The proposed model reflects the fact that expert judgments are imprecise and unreliable in nature.

It is worth noticing that most obtained expressions for the first-order model are given in the explicit form and they do not depend on the sample space of the considered random variable. Moreover, they are identical for continuous and discrete random variables.

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